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Probabilistic Calculations on the Ground State of the Harmonic Oscillator

L. C. Thomas[1]

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A comparison is made between the approaches of Nelson and of Lewis and Davies to quantum probability, using calculations made on the harmonic oscillator. Calculation of the joint distribution of position shows an expected difference in the approaches. The time the particle takes to hit an absorbing counter put in the system is calculated to first order, in both theories, and the results again differ.

KEY WORDS: Probabilistic theories of quantum mechanics; Gaussian instrument; hitting times.

1. INTRODUCTION

Nelson[1,2] suggested that the Schrödinger theory of quantum mechanics is equivalent to one in which it is assumed that every particle of mass m is subject to a Brownian motion whose variance depends inversely on m. This equivalence depends on the belief that in quantum mechanics it is impossible to make repeated measurements of noncommuting observables; in particular, one cannot measure the position of a particle at two different times, and so one cannot calculate the covariance function of position. However, Davies and Lewis[3] have suggested an extension of the usual theory of quantum mechanics in which such measurements are theoretically possible. It is of

[1] University College, Swansea, Great Britain.

822/7/4-1

interest to see how Nelson's theory compares with this extension of quantum mechanics; to do this, we use one of the simplest quantum systems, a one-dimensional quantum oscillator in its ground state. This is represented in Nelson's theory by a particle whose position is described by the Ornstein–Uhlenbeck process with initial Gaussian distribution.

We first calculate the joint distribution of the position of the particle at varying times, but using Gaussian instruments rather than the standard type of position-measuring instruments. In the latter, if the apparatus is set to measure whether the particle is at x_0, then it has probability one of recording the event if the particle is actually at x_0, and probability zero if the particle is not at x_0. A Gaussian instrument, of width σ, measuring the same event has probability $(2\pi\sigma^2)^{-1/2}\exp[-(x-x_0)^2/2\sigma^2]\,dx$ of recording the event if the particle is in the region $(x, x+dx)$. As $\sigma \to 0$, these instruments, suggested first in Ref. 4 and generalized in Ref. 5, tend to standard measuring instruments. Such Gaussian instruments are good approximations of real measurements, since it is not possible to measure exactly a point in a continuum.

In Section 3, a small counter is introduced into the system and the counter records the time at which the particle hits it and is absorbed. It is theoretically possible to calculate the hitting time probabilities in both theories, but for ease of wroking, it is assumed that the absorption rate of the counter is small, of order ϵ. This represents a counter where there is a strong possibility of the particle passing right through the counter without being absorbed. In both the calculations of joint distributions and hitting times, the two approaches give different results. This is somewhat surprising in the case of hitting times, as there appears to be only one measurement involved, which would normally imply agreement of the two theories.

All these calculations can be made for a wider class of quantum systems and the results of the two theories compared. The results indicate that Nelson's theory does not agree with that of Lewis and Davies, and display how unlike classical probability is the probability of quantum theory.

2. JOINT DISTRIBUTIONS OF POSITION

Consider a quantum particle of mass $m = 1$ performing a simple harmonic motion in one dimension under a force $-w^2\mathbf{x}$, with initial state

$$\Psi(x, 0) = (w/\hbar\pi)^{1/4}\exp(-wx^2/2\hbar) \tag{1}$$

The development of the system is given by Schrödinger's equation

$$i\hbar\frac{\partial\Psi}{\partial t} = H\Psi = -\frac{\hbar^2}{2}\frac{\partial^2\Psi}{\partial x^2} + \frac{w^2x^2}{2}\Psi \tag{2}$$

so that at time t, $\Psi(x, t) = \Psi(x, 0)\, e^{-iwt/2}$. Thus the probability density of the position of the particle at time t, $|\Psi(x, t)|^2$, is Gaussian with mean zero and variance $\hbar/2w$.

In Nelson's theory the position of a particle is given by the Markov process $x(t)$ which satisfies the stochastic differential equation

$$dx(t) = b[x(t), t]\, dt + dw_t \tag{3}$$

where w_t is the Wiener process with diffusion constant $\hbar/2$. The initial distribution and $b(x, t)$ are obtained from the quantum system to be described; the initial distribution is defined by $\rho(x, 0) = |\Psi(x, 0)|^2$ and if $\Psi(x, t) = \exp[R(x, t) + iS(x, t)]$, where R and S are real functions, then

$$b(x, t) = \hbar[\partial R/\partial x) + (\partial S/\partial x)] \tag{4}$$

In this case $b(x, t) = -wx$, so the process must satisfy the relation

$$dx(t) = -wx(t)\, dt + dw_t \tag{5}$$

and have an initial Gaussian distribution with mean zero and variance $\hbar/2w$. Such a solution is the Ornstein–Uhlenbeck process, which is a stationary Gaussian Markov process with mean zero and covariance

$$\mathrm{Exp}[X(t)\, X(t + \tau)] = (\hbar/2w) \exp(-w\,|\,\tau\,|).$$

Measuring position with a Gaussian instrument of width σ, Nelson's theory gives

$$\begin{aligned} p(x, t, \sigma) &= \text{probability density that Gaussian instrument placed at} \\ &\qquad x \text{ records particle at time } t \\ &= \left(\frac{1}{2\pi\sigma^2}\right)^{1/2} \int_{-\infty}^{+\infty} \exp\left[-\frac{(x-y)^2}{2\sigma^2}\right] P(y, t)\, dy \end{aligned} \tag{6}$$

where $P(y, t)$ is the probability density of the Ornstein–Uhlenbeck process, i.e., $(w/\hbar\pi)^{1/2} \exp(-wx^2/\hbar)$. Therefore

$$p(x, t, \sigma) = \{\pi[2\sigma^2 + (\hbar/w)]\}^{-1/2} \exp[-wx^2/(2w\sigma^2 + \hbar)] \tag{7}$$

Using the terminology of Ref. 3 to describe the Davies–Lewis theory, the state space (V, τ) is given by $V = \mathscr{T}_s(L^2(R))$, the ordered Banach space of self-adjoint trace class operators on $L^2(R)$, and τ is the trace. As described in Refs. 4 and 5, a Gaussian position-measuring instrument is described by a map $\mathscr{E}^\sigma : V \times B(R) \to V$, where $B(R)$ are the Borel sets of R. The map $\mathscr{E}^\sigma$ is defined by

$$\mathscr{E}^\sigma(M)\, \rho = \int_M \alpha^\sigma(a - Q)\, \rho\alpha^\sigma(a - Q)^*\, da, \qquad M \in B(R), \quad \rho \in V \tag{8}$$

where, if

$$Q = \int_{-\infty}^{\infty} \lambda E^Q(d\lambda)$$

then

$$\alpha^\sigma(a - Q) = (2\pi\sigma^2)^{-1/4} \int_{-\infty}^{+\infty} \{\exp[-(a - \lambda)^2/4\sigma^2]\} E^Q(d\lambda) \tag{9}$$

Thus if we wish to calculate the probability density in this theory, we consider the Borel set $[x_0, x_0 + dx]$, so with abuse of notation,

$$\mathscr{E}^\sigma(x_0, \rho)\, dx = \mathscr{E}^\sigma([x_0, x_0 + dx])\, \rho = \alpha^\sigma(x_0 - Q)\, \rho\alpha^\sigma(x_0 - Q)^*\, dx$$

The probability density that the instrument located at x_0 records the particle at time t then is

$$p_{\mathrm{LD}}(x, t, \sigma)\, dx = \langle \tau, \mathscr{E}^\sigma([x_0, x_0 + dx])(U_t\rho Ut^*)\rangle / \langle \tau, \rho\rangle \tag{10}$$

where $U_t = e^{-iHt/\hbar}$ and $\rho = P_\Psi = |\, 0\rangle\langle 0\,|$, the projection operator onto the subspace spanned by the ground state of the harmonic oscillator. Since for Q the position observable $(E^Q(\lambda)f)(x) = \chi_{[\lambda, -\infty)}(x) f(x)$,

$$\begin{aligned} p_{\mathrm{LD}}(x, t, \sigma) &= \mathrm{tr}[\alpha^\sigma(x_0 - Q)\, U(t)\, P_\Psi U(t)^*\, \alpha^\sigma(x_0 - Q)^*] \\ &= \{\pi[2\sigma^2 + (\hbar/w)]\}^{-1/2} \exp[-wx^2/(2w\sigma^2 + \hbar)] \end{aligned} \tag{11}$$

The two theories agree in calculating the probability of the position of the particle at one given time.

To calculate the joint densities, we measure the position at time t_1 with a Gaussian instrument of width σ_2. Nelson's theory leads to the joint density

$$\begin{aligned} P_N(x_2, t_2, \sigma_2; x_1, t_1, \sigma_1) = {} & (4\pi^3\sigma_1^2\sigma_2^2\hbar/w)^{1/2} \\ & \times \int_{-\infty}^{+\infty} \exp[-(x - x_1)^2/2\sigma_1^2] \exp(-wx^2/\hbar)\, dx \\ & \times \int_{\infty}^{\infty} p_t(x, dy) \exp[-(y - x_2)^2/2\sigma_2^2] \end{aligned} \tag{12}$$

where $p_t(x, dy)$, the transition probability from x to dy in time t of the Ornstein–Uhlenbeck process, is

$$p_t(x, dy) = [w^{-1}\pi\hbar(1 - e^{-2wt})]^{-1/2} \exp[- w(y - e^{-wt}x)^2/\hbar(1 - e^{-2wt})]\, dy \tag{13}$$

Then (12) defines a valid joint density and the marginal distribution obtained by integrating over x_1 is of course Gaussian with mean zero and variance $\sigma_2^2 + (\hbar/2w)$.

In the other theory the equivalent probability density is given by

$$p_{\mathrm{LD}}(x_2, t_2, \sigma_2; x_1, t_1, \sigma_1) = \langle \tau, \mathscr{E}^{\sigma_2}(x_2, U_{t_2-t_1}\mathscr{E}^{\sigma_1}(x_1, U_{t_1}\rho U_{t_1}^*)\, U_{t_2-t_1}^*)\rangle / \langle \tau, \rho \rangle$$

$$= \sum_{m,n=0}^{\infty} \{\exp[i(t_2 - t_1)\, w(m - n)]\}(4\pi^2\sigma_1{}^2\sigma_2{}^2)^{-1/2}$$

$$\times \int_{-\infty}^{+\infty} h_n(p)\, h_m(p) \exp[-(p - x_2)^2/2\sigma_2{}^2]\, dp$$

$$\times \int_{-\infty}^{+\infty} h_n(q)\, h_0(q) \exp[-(q - x_1)^2/4\sigma_1{}^2]\, dq$$

$$\times \int_{-\infty}^{+\infty} h_m(r)\, h_0(r) \exp[-(r - x_1)^2/4\sigma_1{}^2]\, dr \quad (14)$$

where h_n is the nth eigenfunction of the harmonic oscillator. This is not the same distribution as (12), as can be see by calculating the marginal distribution$\int p(x_2, t_2, \sigma_2; x_1, t_1, \sigma_1)\, dx_1$ which is Gaussian with mean zero and variance $\sigma_2{}^2 + (\hbar/2w) + (\hbar^2/4w^2\sigma_1{}^2) \sin^2[(t_2 - t_1)w]$. This marginal distribution is the same as measuring if the particle is anywhere in R at time t_1, followed by measuring if it is at x_2 at t_2. This could be done by introducing a screen with an infinite hole in it at time t_1—surely equivalent to no screen at all—and yet if there were no measurement at time t_1, the distribution would be different. This paradox is a consequence of the requirement that a measurement of a quantum state immediately changes the state, no matter how little information is obtained from the measurement. Thus it is hardly surprising that a theory based on classical probability does not incorporate such vagaries.

3. HITTING TIMES

For the second comparison of the two theories, we consider the model of a ground-state harmonic oscillator interacting with an absorbing particle detector. This particle detector covers a small area E, and when the particle hits it the particle is absorbed and the hitting time noted. Such a problem can be treated in the Lewis–Davies theory by using the framework of quantum stochastic processes.[6] Since there is a change in the number of particles in the system—from one to zero—the state space (V, τ) is taken as $V = \mathscr{T}_s(\mathscr{H})$, where $\mathscr{H} = \mathscr{F}(L^2(R))$, the symmetric Fock space built on $L^2(R)$. The counter is represented by the canonical annihilation operator $A(f)$, f being an L^2 function with support in E. If H is the Hamiltonian of the harmonic oscillator on $L^2(R)$, $\mathscr{F}(H)$ is its extension to Fock space.

A quantum stochastic process, as defined in Ref. 6, is a set of instruments $\mathscr{E}^t$, indexed by a time parameter $t \in [0, \infty)$, each of which maps the initial state of the system onto the state at time t, dependent upon what has happened up to time t. For these instruments, which measure when the particle hits the counter, the sample space X_t of $\mathscr{E}^t$ consists of all possible times the hitting could occur together with a point z, representing the possibility that no hitting has occurred. Thus $X_t = \{s : s \in [0, t]\} \cup z$. To define such instruments, we use the facts proved in Ref. 6, that $S_t\rho = \mathscr{E}^t(z)\,\rho$, the states conditional upon no hitting, form a semigroup, and if $S_t\rho = B_t\rho B_t{}^*$, then this no hitting time development is a perturbation of the usual time development given by

$$B_t = \exp[\mathscr{F}(-i\hbar^{-1}Ht) - \tfrac{1}{2}A^*(f)\,A(f)t] \tag{15}$$

For the other values of X_t we simply use the fact that on hitting the counter, the state of the particle changes from ρ to $A(f)\,\rho A(f)^*$ so that

$$\mathscr{E}^t(s)\,\rho = B_{t-s}A(f)\,B_s\rho B_s{}^*A(f)^*\,B_{t-s}^* \tag{16}$$

The initial state of the system is $\rho = \boldsymbol{\Psi} \otimes \boldsymbol{\Psi} = P_{\boldsymbol{\Psi}}$, where $\boldsymbol{\Psi}$ has Fock coordinates $(0, \Psi(x, 0), 0, 0,...)$, Ψ defined in Eq. (1). Thus the state $A(f)\,B_s\rho B_s{}^*A(f)^*$ has only a component in the first Fock coordinate and B_{t-s} has no effect on it. Thus (16) can be written as $A(f)\,B_s\rho B_s{}^*A(f)^*$.

If ξ is the random variable denoting the hitting time of the particle, it follows from Ref. 6 that the probability of the hitting times is

$$\begin{aligned} P(t \leqslant \xi_{\mathrm{LD}} \leqslant t + dt) &= [\langle\tau, \mathscr{E}^t(t)\,\rho\rangle/\langle\tau, \rho\rangle]\,dt + o(dt) \\ &= [\langle\tau, A(f)\,B_t\rho B_t{}^*A(f)^*\rangle/\langle\tau, \rho\rangle]\,dt + o(dt) \end{aligned} \tag{17}$$

In most cases this expression is difficult to calculate, so assume the detector has a small interaction rate, i.e., there is a strong possibility the particle will pass through the detector without triggering it off and being absorbed, and so only slightly perturbs the system. Let ϵ be small and let the state change on being absorbed from ρ to $\epsilon A(f)\,\rho A(f)^*$; then B_t also changes to $\exp[\mathscr{F}(-i\hbar^{-1}Ht) - \frac{1}{2}\epsilon A(f)^*\,A(f)\,t]$. Substituting in (17) gives the new hitting probability, which, to first order in ϵ, is

$$\begin{aligned} &P(t \leqslant \xi_{\mathrm{LD}} \leqslant t + dt) \\ &\quad = \epsilon(w/\hbar\pi)^{1/2}\left\{\int_E [\exp(-x^2w/\hbar)]\,f(x)\,dx\right\}^2 dt + o(\epsilon) + o(dt) \\ &\quad = \epsilon\,|\langle f, \Psi\rangle|^2\,dt + o(\epsilon) + o(dt) \end{aligned} \tag{18}$$

where Ψ is defined in Eq. (1).

For Nelson's theory we require the probability that, given the particle is at x at time t, it will hit the counter in the interval $[t, t + dt]$. Let this be $k(x)\,dt + o(dt)$. The quantum state corresponding to a particle being definitely at x is approximately P_Ψ, where $\Psi(y) = \delta(x - y)$. If we take $B_t \rho B_t^*$ to be of this form and substitute in (17), we are led to the assumption that $k(x) = \epsilon\, |f(x)|^2$. This argument cannot be made valid, but we shall take this equation as the connection between the two theories and use it to calculate the hitting probability in Nelson's model. Following Dynkin,[7] if the probability density $p(x, t)$ of the system without the counter satisfies the Fokker–Planck equation

$$\partial p/\partial t = G^* p \tag{19}$$

then if $P(x, t)$ is the probability density for the process including the killing counter, it satisfies

$$\partial P/\partial t = G^* P - k(x)\, P \tag{20}$$

For the Ornstein–Uhlenbeck process of Section 2,

$$G^* = \tfrac{1}{2}\hbar(\partial^2/\partial x^2) + wx(\partial/\partial x) + w,$$

and the hitting probability is given by

$$P(t \leqslant \xi_N \leqslant t + dt) = \int_{-\infty}^{+\infty} [k(x)\, dt + o(dt)]\, P(x, t)\, dx \tag{21}$$

Since $k(x) = \epsilon\, |f(x)|^2$, $P(x, t)$ must satisfy

$$\frac{\partial P}{\partial t} = \frac{\hbar}{2} \frac{\partial^2 P}{\partial x^2} + wx \frac{\partial P}{\partial x} + wP - \epsilon\, |f(x)|^2\, P \tag{22}$$

with initial conditions $P(x, 0) = |\Psi(x, 0)|^2 = (w^{-1}\pi\hbar)^{-1/2} \exp(-wx^2/\hbar)$. Expanding P in terms of ϵ, we get, using Eq. (1),

$$P(x, t) = |\Psi(x, 0)|^2[1 + \epsilon g(x, t) + o(\epsilon)] \tag{23}$$

Equations (21) and (23) give

$$\begin{aligned} &P(t \leqslant \xi_N \leqslant t + dt) \\ &\quad = \left\{\epsilon (w/\pi\hbar)^{1/2} \int_E [\exp(-x^2 w/\hbar)]\, |f(x)|^2\, dx\right\} dt + o(\epsilon) + o(dt) \\ &\quad = \epsilon \langle |\Psi|^2, |f|^2 \rangle\, dt + o(\epsilon) + o(dt) \end{aligned} \tag{24}$$

Thus the two theories vary even in first order in ϵ. This is somewhat surprising as the theories should agree if only one measurement is made. However, the introduction of an absorbing counter is treated differently in quantum mechanics and in probability theory and this causes the difference in results. If the detector function is changed from $f(x)$ to $e^{i\lambda(x)} f(x)$ for some real function

λ, then this will change the first order results in the Lewis–Davies theory but not in the Nelson case. Such a change in the detector function could be thought of as giving the detector a preference for a particle with a certain momentum and so it appears that in the Lewis–Davies case the detector is momentum dependent while in the Nelson case it depends only on the position of the particle.

Since it is feasible that hitting times could actually be measured for some physical systems, it is possible that this work could be extended to give results that could be checked experimentally.

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